Journal of Pure and Applied Algebra 213 (2009) 649–663

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

journal homepage: www.elsevier.com/locate/jpaa

# Invariant theory and reversible-equivariant vector fields

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ARTICLE INFO

*Article history:*
Received 9 October 2007
Received in revised form 28 April 2008
Available online 9 September 2008
Communicated by E.M. Friedlander

*MSC:*
35B32
37G40

ABSTRACT

In this paper we present results for the systematic study of reversible-equivariant vector fields – namely, in the simultaneous presence of symmetries and reversing symmetries – by employing algebraic techniques from invariant theory for compact Lie groups. The Hilbert–Poincaré series and their associated Molien formulae are introduced, and we prove the character formulae for the computation of dimensions of spaces of homogeneous anti-invariant polynomial functions and reversible-equivariant polynomial mappings. A symbolic algorithm is obtained for the computation of generators for the module of reversible-equivariant polynomial mappings over the ring of invariant polynomials. We show that this computation can be obtained directly from a well-known situation, namely from the generators of the ring of invariants and the module of the equivariants.



## 1. Introduction

The conventional notion of presence of symmetries (equivariances) and reversing symmetries in a system of differential equations consists of phase space transformations, including time transformations for reversing symmetries, that leave the equations of motion invariant. The formulation of this situation is given as follows: consider the system

$$\dot{x} = G(x) \tag{1.1}$$

defined on a finite-dimensional vector space $V$ of state variables, where $G : V \to V$ is a smooth vector field. The space $V$ carries an action of a compact Lie group $\Gamma$ together with a distinguished subgroup $\Gamma_+$ such that $\gamma \in \Gamma_+$ maps trajectories onto trajectories of (1.1) with the direction of time being preserved, while $\gamma \in \Gamma \setminus \Gamma_+$ maps trajectories onto trajectories of (1.1) with the direction of time being reversed. Dynamical systems with such property are called *reversible-equivariant systems* and $\Gamma$ is called the *reversing symmetry group* of the ordinary differential equation (1.1). The elements of the subgroup $\Gamma_+$ act as *spatial symmetries* or simply *symmetries* and the elements of the subset $\Gamma_- = \Gamma \setminus \Gamma_+$ act as *time-reversing symmetries* or simply *reversing symmetries*.

Let $(\rho, \Gamma)$ denote $V$ under the representation $\rho$ of $\Gamma$. The requirement of $G$ being *reversible-equivariant* with respect to $\Gamma$ is

$$G(\rho(\gamma)x) = \sigma(\gamma)\rho(\gamma)\,G(x), \tag{1.2}$$

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0022-4049/$ – see front matter 
doi:10.1016/j.jpaa.2008.08.002

for all $\gamma \in \Gamma$ and $x \in V$, where

$$\sigma : \Gamma \to \mathbf{Z}_2 = \{\pm 1\} \tag{1.3}$$

is the Lie group homomorphism which is 1 on the subgroup $\Gamma_+$ of the symmetries and $-1$ on the subset $\Gamma_-$ of the reversing symmetries. We note that $\Gamma_+ = \ker(\sigma)$, so it is a normal subgroup of $\Gamma$ of index 2.

The starting point for local or global analysis of systems under symmetries is to find the general form of the vector field $G$ in (1.1) that satisfies the symmetry constraints.

In the equivariant case, the theorems by Schwarz and Poénaru (Golubitsky et al. [8, Theorem XII 4.3 and Theorem XII 5.2]) reduce this task to a purely algebraic problem in invariant theory. We find a set of generators for the ring $\mathcal{P}_V(\Gamma)$ of invariant polynomial functions $V \to \mathbf{R}$ and for the module $\vec{\mathcal{P}}_V(\Gamma)$ of all equivariant polynomial mappings $V \to V$. What makes it feasible through computational methods is the simple observation that $\mathcal{P}_V(\Gamma)$ and $\vec{\mathcal{P}}_V(\Gamma)$ are graded algebras by the polynomial degree, that is,

$$\mathcal{P}_V(\Gamma) = \bigoplus_{d=0}^{\infty} \mathcal{P}_V^d(\Gamma) \quad \text{and} \quad \vec{\mathcal{P}}_V(\Gamma) = \bigoplus_{d=0}^{\infty} \vec{\mathcal{P}}_V^d(\Gamma),$$

where $\mathcal{P}_V^d(\Gamma)$ is the space of homogeneous polynomial invariants of degree $d$ and $\vec{\mathcal{P}}_V^d(\Gamma)$ is the space of equivariant mappings with homogeneous polynomial components of degree $d$. Since these are finite-dimensional vector spaces, generating sets may be found by Linear Algebra together with the Hilbert–Poincaré series and their associated Molien formulae. See Sturmfels [16] for example, or Gatermann [7], where the tools of computational invariant theory are developed with the view towards their applications to equivariant bifurcation theory. The symbolic computation packages GAP [5] and Singular, [10] have all these tools implemented in their libraries. In the reversible-equivariant case we face a similar situation. The space $\vec{\mathcal{Q}}_V(\Gamma)$ of reversible-equivariant polynomial mappings is a finitely generated graded module over the ring $\mathcal{P}_V(\Gamma)$ and so the same methods described before could be adapted to work in this context.

In this paper we follow a different approach, based on a link existent between the invariant theory for $\Gamma$ and for its normal subgroup $\Gamma_+$. In order to provide this link, we observe that $\mathcal{P}_V(\Gamma)$ is a subring of $\mathcal{P}_V(\Gamma_+)$ and so this may be regarded as a module over $\mathcal{P}_V(\Gamma)$. Next we introduce the space $\mathcal{Q}_V(\Gamma)$ of anti-invariant polynomial functions: a polynomial function $f : V \to \mathbf{R}$ is called *anti-invariant* if

$$f(\rho(\gamma)x) = \sigma(\gamma)f(x), \tag{1.4}$$

for all $\gamma \in \Gamma$ and $x \in V$. We have that $\mathcal{Q}_V(\Gamma)$ is a finitely generated graded module over $\mathcal{P}_V(\Gamma)$. Now, by means of the *relative Reynolds operators*, we obtain our first main result which states that there are decompositions

$$\mathcal{P}_V(\Gamma_+) = \mathcal{P}_V(\Gamma) \oplus \mathcal{Q}_V(\Gamma) \quad \text{and} \quad \vec{\mathcal{P}}_V(\Gamma_+) = \vec{\mathcal{P}}_V(\Gamma) \oplus \vec{\mathcal{Q}}_V(\Gamma)$$

as direct sums of modules over $\mathcal{P}_V(\Gamma)$.

These decompositions are the basis for our second main result, namely the algorithm that computes a generating set for reversible-equivariant polynomial mappings as a module over the ring of invariant polynomial functions. The procedure uses relative Reynolds operators, a Hilbert basis of $\mathcal{P}_V(\Gamma_+)$ and a generating set of the module $\vec{\mathcal{P}}_V(\Gamma_+)$. We note that these two sets can be readily obtained by the standard methods of computational invariant theory applied to $\Gamma_+$. It should be pointed out that it is not evident, a priori, that by applying the Reynolds operator to a Hilbert basis of $\mathcal{P}_V(\Gamma_+)$ one produces a generating set of $\mathcal{Q}_V(\Gamma)$ as a module over the ring $\mathcal{P}_V(\Gamma)$. In fact, it is remarkable that such a simple prescription gives a non-trivial procedure to obtain a generating set of $\mathcal{Q}_V(\Gamma)$ from a Hilbert basis of $\mathcal{P}_V(\Gamma_+)$.

We have also included here a brief presentation of the Hilbert–Poincaré series and their associated Molien formulae. In addition, we prove character formulae for the anti-invariants and reversible-equivariants. Although these tools are not used in the proofs of the main results, we felt that, on the one hand, it is instructive to show that the standard concepts of invariant theory can be adapted to the reversible-equivariant context; on the other hand, it is convenient for future references to have all this material collected in one place.

## 2. The structure of anti-invariants and reversible-equivariants

We start this section with some basic facts about the reversible-equivariant theory. Next we obtain results about the structure of the spaces of anti-invariant functions and reversible-equivariant mappings. Finally, we generalize, for these spaces, the character formulae given by Sattinger [13, Theorem 5.10] for the dimensions of the subspaces of homogeneous polynomials of each fixed degree.

### 2.1. The general setting

Let $\Gamma$ be a reversing symmetry group and $\sigma$ the non-trivial homomorphism (1.3). We choose $\delta \in \Gamma_-$ to write $\Gamma$ as a disjoint union of left-cosets:

$$\Gamma = \Gamma_+ \dot{\cup} \Gamma_- = \Gamma_+ \dot{\cup} \; \delta\, \Gamma_+.$$

If $(\eta, W)$ is also a representation of $\Gamma$, let us denote by $\vec{\mathcal{P}}_{V,W}(\Gamma)$ the $\mathcal{P}_V(\Gamma)$-module of equivariant polynomial mappings $G : V \to W$,

$$G(\rho(\gamma)x) = \eta(\gamma)G(x),$$

for all $\gamma \in \Gamma$ and $x \in V$. When $(\eta, W) = (\rho, V)$, this is $\vec{\mathcal{P}}_V(\Gamma)$ defined in Section 1.

We proceed by introducing the dual of a representation: given the homomorphism (1.3), the $\sigma$*-dual* of the representation $\rho$ is the representation $\rho_\sigma$ of $\Gamma$ on $V$ defined by

$$\begin{aligned} \rho_\sigma : \quad & \Gamma \longrightarrow \mathbf{GL}(V) \\ & \gamma \mapsto \sigma(\gamma)\rho(\gamma). \end{aligned}$$

Notice that $(\rho_\sigma)_\sigma = \rho$.

A representation $(\rho, V)$ of $\Gamma$ is said to be *self-dual* if it is $\Gamma$-isomorphic to $(\rho_\sigma, V)$ or, equivalently, if there exists a reversible-equivariant linear isomorphism on $V$. In this case, we say that $V$ is a *self-dual* vector space.

**Remark 2.1.** Using the $\sigma$-dual representation $\rho_\sigma$ of $\rho$, condition (1.2) may be written as

$$G(\rho(\gamma)x) = \rho_\sigma(\gamma)G(x).$$

In this way, we may regard a reversible-equivariant mapping on $V$ as an equivariant mapping from $(\rho, V)$ to $(\rho_\sigma, V)$. Also, equation (1.4) is equivalent to the requirement that $f$ is an equivariant mapping from $(\rho, V)$ to $(\sigma, \mathbf{R})$. Therefore, the existence of a finite generating set for $\mathcal{Q}_V(\Gamma)$ and for $\vec{\mathcal{Q}}_V(\Gamma)$ is guaranteed by Poénaru's Theorem (Golubitsky et al. [8, Theorem XII 5.2]). ◇

**Remark 2.2.** We make the following two observations:

(i) There is an important particular case in the reversible-equivariant theory, i. e., when $\Gamma$ is a two-element group. This is the *purely reversible* framework, where there are no non-trivial symmetries and only one non-trivial reversing symmetry which is an involution.

(ii) When $\sigma$ in (1.3) is the trivial homomorphism, $(\rho, V)$ and $(\rho_\sigma, V)$ are the same representation and so we encounter the *purely equivariant* framework. ◇

*2.2. The application of the invariant theory*

In this subsection we relate the rings of invariant polynomial functions of $\Gamma$ and $\Gamma_+$, the module of anti-invariants playing a fundamental role in this construction. We start by observing that $\mathcal{P}_V(\Gamma_+)$ (as well as $\mathcal{Q}_V(\Gamma)$) is a module over $\mathcal{P}_V(\Gamma)$.

Let us consider the *relative Reynolds operator* from $\Gamma_+$ to $\Gamma$ on $\mathcal{P}_V(\Gamma_+)$, $R^\Gamma_{\Gamma_+} : \mathcal{P}_V(\Gamma_+) \to \mathcal{P}_V(\Gamma_+)$. In our particular case, it is simply given by

$$R^\Gamma_{\Gamma_+}(f)(x) = \frac{1}{2}\sum_{\gamma\Gamma_+} f(\gamma x) = \frac{1}{2}\left(f(x) + f(\delta x)\right), \tag{2.1}$$

for an arbitrary (and fixed) $\delta \in \Gamma_-$.

Now we define the *relative Reynolds* $\sigma$-operator on $\mathcal{P}_V(\Gamma_+)$, $S^\Gamma_{\Gamma_+} : \mathcal{P}_V(\Gamma_+) \to \mathcal{P}_V(\Gamma_+)$, by

$$S^\Gamma_{\Gamma_+}(f)(x) = \frac{1}{2}\sum_{\gamma\Gamma_+} \sigma(\gamma) f(\gamma x).$$

So

$$S^\Gamma_{\Gamma_+}(f)(x) = \frac{1}{2}\left(f(x) - f(\delta x)\right), \tag{2.2}$$

for an arbitrary (and fixed) $\delta \in \Gamma_-$.

Let us denote by $I_{\mathcal{P}_V(\Gamma_+)}$ the identity map on $\mathcal{P}_V(\Gamma_+)$. We then have:

**Proposition 2.3.** *The relative Reynolds operator and* $\sigma$*-operator* $R^\Gamma_{\Gamma_+}, S^\Gamma_{\Gamma_+}$ *satisfy the following properties:*

(i) *They are homomorphisms of* $\mathcal{P}_V(\Gamma)$*-modules and*

$$R^\Gamma_{\Gamma_+} + S^\Gamma_{\Gamma_+} = I_{\mathcal{P}_V(\Gamma_+)}. \tag{2.3}$$

(ii) *They are idempotent projections with*

$$\ker(R^\Gamma_{\Gamma_+}) = \mathcal{Q}_V(\Gamma) \quad \textit{and} \quad \ker(S^\Gamma_{\Gamma_+}) = \mathcal{P}_V(\Gamma),$$

$$\operatorname{im}(R^\Gamma_{\Gamma_+}) = \mathcal{P}_V(\Gamma) \quad \textit{and} \quad \operatorname{im}(S^\Gamma_{\Gamma_+}) = \mathcal{Q}_V(\Gamma).$$

(iii) *The following decompositions (as direct sum of $\mathcal{P}_V(\Gamma)$-modules) hold:*

$$\mathcal{P}_V(\Gamma_+) = \ker(R^{\Gamma}_{\Gamma_+}) \oplus \operatorname{im}(R^{\Gamma}_{\Gamma_+}) = \ker(S^{\Gamma}_{\Gamma_+}) \oplus \operatorname{im}(S^{\Gamma}_{\Gamma_+}).$$

**Proof.** Item (i) is straightforward. For (ii) and (iii), the statements referring to the operator $R^{\Gamma}_{\Gamma_+}$ can be found in Neusel [12, p. 103]. Moreover, these proofs can be easily modified in order to obtain the analogous results for the operator $S^{\Gamma}_{\Gamma_+}$. □

**Corollary 2.4.** *The following direct sum decomposition of modules over the ring $\mathcal{P}_V(\Gamma)$ holds:*

$$\mathcal{P}_V(\Gamma_+) = \mathcal{P}_V(\Gamma) \oplus \mathcal{Q}_V(\Gamma).$$

Now we relate the modules of equivariant polynomial mappings under $\Gamma$ and $\Gamma_+$. This can be done by a similar construction as above for the rings of invariant polynomial functions under $\Gamma$ and $\Gamma_+$: we consider the relative Reynolds operator from $\Gamma_+$ to $\Gamma$ on $\vec{\mathcal{P}}_V(\Gamma_+)$, $\vec{R}^{\Gamma}_{\Gamma_+} : \vec{\mathcal{P}}_V(\Gamma_+) \to \vec{\mathcal{P}}_V(\Gamma_+)$, which is, in the present case, given by

$$\vec{R}^{\Gamma}_{\Gamma_+}(G)(x) = \frac{1}{2}\sum_{\gamma\Gamma_+} \gamma^{-1}G(\gamma x) = \frac{1}{2}\left(G(x) + \delta^{-1}G(\delta x)\right), \tag{2.4}$$

for an arbitrary $\delta \in \Gamma_-$.

Now we define the *relative Reynolds $\sigma$-operator* from $\Gamma_+$ to $\Gamma$ on $\vec{\mathcal{P}}_V(\Gamma_+)$, $\vec{S}^{\Gamma}_{\Gamma_+} : \vec{\mathcal{P}}_V(\Gamma_+) \to \vec{\mathcal{P}}_V(\Gamma_+)$, by

$$\vec{S}^{\Gamma}_{\Gamma_+}(G)(x) = \frac{1}{2}\sum_{\gamma\Gamma_+} \sigma(\gamma)\gamma^{-1}G(\gamma x).$$

So

$$\vec{S}^{\Gamma}_{\Gamma_+}(G)(x) = \frac{1}{2}\left(G(x) - \delta^{-1}G(\delta x)\right), \tag{2.5}$$

for an arbitrary $\delta \in \Gamma_-$.

Let us denote by $I_{\vec{\mathcal{P}}_V(\Gamma_+)}$ the identity map on $\vec{\mathcal{P}}_V(\Gamma_+)$. We then have:

**Proposition 2.5.** *The Reynolds operator $\vec{R}^{\Gamma}_{\Gamma_+}$ and the Reynolds $\sigma$-operator $\vec{S}^{\Gamma}_{\Gamma_+}$ satisfy the following properties:*

(i) *They are homomorphisms of $\mathcal{P}_V(\Gamma)$-modules and*

$$\vec{R}^{\Gamma}_{\Gamma_+} + \vec{S}^{\Gamma}_{\Gamma_+} = I_{\vec{\mathcal{P}}_V(\Gamma_+)}. \tag{2.6}$$

(ii) *They are idempotent projections with*

$$\ker(\vec{R}^{\Gamma}_{\Gamma_+}) = \vec{\mathcal{Q}}_V(\Gamma) \quad \textit{and} \quad \ker(\vec{S}^{\Gamma}_{\Gamma_+}) = \vec{\mathcal{P}}_V(\Gamma),$$

$$\operatorname{im}(\vec{R}^{\Gamma}_{\Gamma_+}) = \vec{\mathcal{P}}_V(\Gamma) \quad \textit{and} \quad \operatorname{im}(\vec{S}^{\Gamma}_{\Gamma_+}) = \vec{\mathcal{Q}}_V(\Gamma).$$

(iii) *The following decompositions (as direct sum of $\mathcal{P}_V(\Gamma)$-modules) hold:*

$$\vec{\mathcal{P}}_V(\Gamma_+) = \ker(\vec{S}^{\Gamma}_{\Gamma_+}) \oplus \operatorname{im}(\vec{S}^{\Gamma}_{\Gamma_+}) = \ker(\vec{R}^{\Gamma}_{\Gamma_+}) \oplus \operatorname{im}(\vec{R}^{\Gamma}_{\Gamma_+}).$$

**Proof.** It is analogous to the proof of Proposition 2.3. □

We are now in a position to state the following corollary which, together with Corollary 2.4, forms our first main result.

**Corollary 2.6.** *The following direct sum decomposition of modules over the ring $\mathcal{P}_V(\Gamma)$ holds:*

$$\vec{\mathcal{P}}_V(\Gamma_+) = \vec{\mathcal{P}}_V(\Gamma) \oplus \vec{\mathcal{Q}}_V(\Gamma).$$

We end this subsection with a remark.

**Remark 2.7.** The notion of anti-invariant polynomial function is a particular case of the definition of *relative invariant* of a group $\Gamma$. Stanley [15] carried out a study of the structure of relative invariants when $\Gamma$ is a finite group generated by pseudo-reflections. Smith [14] has focused attention on the invariant theory by using results on relative invariants when $\Gamma$ is also an arbitrary finite group. Let us switch to the notation of Smith. Let $\mathbb{F}$ be a field, $\rho : \Gamma \to \mathbf{GL}(n, \mathbb{F})$ a representation of a finite group $\Gamma$ and $\chi : \Gamma \to \mathbb{F}^{\times}$ be a linear character. Therein, the ring of invariants of $\Gamma$ is denoted by

$$\mathbb{F}[V]^{\Gamma} = \{f \in \mathbb{F}[V] : f(\gamma x) = f(x), \forall \gamma \in \Gamma\}$$

and the module of $\chi$-*relative invariants* of $\Gamma$ by

$$\mathbb{F}[V]^{\Gamma}_{\chi} = \{f \in \mathbb{F}[V] : f(\gamma x) = \chi(\gamma) f(x), \forall \gamma \in \Gamma\}.$$

Now let $\Sigma$ be a subgroup of $\Gamma$ of index $m = [\Gamma : \Sigma]$. Then,

$$\mathbb{F}[V]^{\Sigma} = \mathbb{F}[V]^{\Gamma} \oplus \mathbb{F}[V]^{\Gamma}_{\chi} \oplus \cdots \oplus \mathbb{F}[V]^{\Gamma}_{\chi^{m-1}}. \tag{2.7}$$

If $\gamma_\chi \Sigma$ generates the cyclic group $\Gamma/\Sigma$, then the subspaces $\mathbb{F}[V]^{\Gamma}_{\chi^j}$, for $j = 0, \ldots, m-1$, are the eigenspaces associated with the eigenvalues $\chi(\gamma_\chi)^j$ of the action of $\gamma_\chi$ on $\mathbb{F}[V]^{\Sigma}$. In particular, when $\Gamma$ is generated by pseudo-reflections, Smith [14, Theorem 2.7] shows that $\mathbb{F}[V]^{\Gamma}_{\chi}$ is a free module over $\mathbb{F}[V]^{\Gamma}$ on a single generator which can be constructed from the action of $\Gamma$ on $\mathbb{F}^n$. Switching back to our notation, the module $\mathcal{Q}_V(\Gamma)$ of anti-invariants is the module of $\sigma$-relative invariants $\mathbf{R}[V]^{\Gamma}_{\sigma}$. Thus, when $\Gamma$ is a finite group, the ring $\mathcal{P}_V(\Gamma)$ and the module $\mathcal{Q}_V(\Gamma)$ are the eigenspaces associated to the eigenvalues 1 and $-1$, respectively, of the action of $\delta \in \Gamma_-$ on $\mathcal{P}_V(\Gamma_+)$. Therefore, decomposition (2.7) is a generalization of Corollary 2.4 (where $m = 2$ and $\chi$ is a real-valued linear character). On the other hand, Corollary 2.4 is a natural extension of the decomposition (2.7) for arbitrary compact Lie groups. ◇

### 2.3. Hilbert–Poincaré series, Molien formulae and character formulae for anti-invariants and reversible-equivariants

As well as for invariants and equivariants, we define below the Hilbert–Poincaré series for the modules $\mathcal{Q}_V(\Gamma)$ and $\vec{\mathcal{Q}}_V(\Gamma)$ and establish Molien formulae for them.

We begin by writing the natural gradings for $\mathcal{Q}_V(\Gamma)$ and $\vec{\mathcal{Q}}_V(\Gamma)$:

$$\mathcal{Q}_V(\Gamma) = \bigoplus_{d=0}^{\infty} \mathcal{Q}^d_V(\Gamma) \quad \text{and} \quad \vec{\mathcal{Q}}_V(\Gamma) = \bigoplus_{d=0}^{\infty} \vec{\mathcal{Q}}^d_V(\Gamma), \tag{2.8}$$

where $\mathcal{Q}^d_V(\Gamma)$ is the space of homogeneous polynomial anti-invariants of degree $d$ and $\vec{\mathcal{Q}}^d_V(\Gamma)$ is the space of reversible-equivariant mappings with homogeneous polynomial components of degree $d$. It follows from Corollaries 2.4 and 2.6 that we have the direct sum decompositions of vector spaces

$$\mathcal{P}^d_V(\Gamma_+) = \mathcal{P}^d_V(\Gamma) \oplus \mathcal{Q}^d_V(\Gamma) \tag{2.9}$$

and

$$\vec{\mathcal{P}}^d_V(\Gamma_+) = \vec{\mathcal{P}}^d_V(\Gamma) \oplus \vec{\mathcal{Q}}^d_V(\Gamma) \tag{2.10}$$

for every degree $d \in \mathbf{N}$.

The Hilbert–Poincaré series for $\mathcal{Q}_V(\Gamma)$ and for $\vec{\mathcal{Q}}_V(\Gamma)$ are defined as the following formal power series:

$$\tilde{\Phi}^{\Gamma}_V(t) = \sum_{d=0}^{\infty} \dim \mathcal{Q}^d_V(\Gamma)\, t^d \quad \text{and} \quad \tilde{\Psi}^{\Gamma}_V(t) = \sum_{d=0}^{\infty} \dim \vec{\mathcal{Q}}^d_V(\Gamma)\, t^d.$$

If we apply the general Molien formula (see Gatermann [6, Theorem 12.2]), we obtain each Hilbert–Poincaré series above in terms of the normalized Haar integral over $\Gamma$:

$$\tilde{\Phi}^{\Gamma}_V(t) = \int_{\Gamma} \frac{\sigma(\gamma)}{\det(1 - t\rho(\gamma))} \mathrm{d}\gamma \quad \text{and} \quad \tilde{\Psi}^{\Gamma}_V(t) = \int_{\Gamma} \frac{\sigma(\gamma)\chi_V(\gamma)}{\det(1 - t\rho(\gamma))} \mathrm{d}\gamma,$$

where $\chi_V$ is the character of $(\rho, V)$.

We now present in Corollary 2.9 the expressions for the dimensions of the homogeneous component of (2.8) in terms of the character function and of the normalized Haar integral over $\Gamma$. See Sattinger [13, Theorem 5.10] or Antoneli et al. [1] for the cases of invariants and of purely equivariants.

Let $\mathcal{P}^d_V$ denote the space of homogeneous polynomial functions of degree $d$ and let $L^d_s(V)$ denote the space of real-valued symmetric $d$-multilinear functions on $V^d$. Then we have a canonical isomorphism

$$L^d_s(V) \cong \mathcal{P}^d_V. \tag{2.11}$$

With the analogue for $W$-valued mappings, we also have

$$L^d_s(V, W) \cong \vec{\mathcal{P}}^d_{V,W}. \tag{2.12}$$

Now, from Goodman et al. [9, p. 621],

$$L^d_s(V, W) \cong \mathrm{Hom}(S^d V, W). \tag{2.13}$$

Since $V^* = \mathrm{Hom}(V, \mathbf{R})$ and $V^* \otimes W \cong \mathrm{Hom}(V, W)$ (see Goodman et al. [9, p. 618]),

$$L_s^d(V, W) \cong \mathrm{Hom}(S^d V, W) \cong (S^d V)^* \otimes W,$$

which, together with (2.12), gives

$$\vec{\mathcal{P}}^d_{V,W} \cong (S^d V)^* \otimes W. \tag{2.14}$$

If $(\rho, V)$ and $(\eta, W)$ are two representations of $\Gamma$, then the group $\Gamma$ acts naturally on the space of $W$-valued mappings $\vec{\mathcal{P}}_{V,W}$ by

$$\gamma \cdot G = \eta(\gamma^{-1})\, G\rho(\gamma),$$

for all $\gamma \in \Gamma$ and $G \in \vec{\mathcal{P}}_{V,W}$. So we have

$$\vec{\mathcal{P}}^d_{V,W}(\Gamma) = \mathrm{Fix}_{\vec{\mathcal{P}}^d_{V,W}}(\Gamma). \tag{2.15}$$

Combining (2.14) with (2.15) we get

$$\vec{\mathcal{P}}^d_{V,W}(\Gamma) \cong \mathrm{Fix}_{(S^d V)^* \otimes W}(\Gamma). \tag{2.16}$$

We then obtain:

**Theorem 2.8.** *Let $\Gamma$ be a compact Lie group. Let $(\rho, V)$ and $(\eta, W)$ be two finite-dimensional representations of $\Gamma$ with corresponding characters $\chi_V$ and $\chi_W$. Then*

$$\dim \vec{\mathcal{P}}^d_{V,W}(\Gamma) = \int_\Gamma \chi_{V(d)}(\gamma)\chi_W(\gamma)\mathrm{d}\gamma,$$

*where $\chi_{V(d)}$ is the character afforded by the induced action of $\Gamma$ on $S^d V$.*

**Proof.** The Trace Formula for fixed-point subspaces (Golubitsky et al. [8, Theorem XIII 2.3]) combined with equation (2.16) leads to

$$\begin{aligned}
\dim \vec{\mathcal{P}}^d_{V,W}(\Gamma) &= \dim \mathrm{Fix}_{(S^d V)^* \otimes W}(\Gamma) \\
&= \int_\Gamma \chi_{(S^d V)^* \otimes W}(\gamma)\mathrm{d}\gamma \\
&= \int_\Gamma \chi_{(S^d V)^*}(\gamma)\chi_W(\gamma)\mathrm{d}\gamma \\
&= \int_\Gamma \chi_{(S^d V)}(\gamma^{-1})\chi_W(\gamma)\mathrm{d}\gamma \\
&= \int_\Gamma \chi_{V(d)}(\gamma)\chi_W(\gamma)\mathrm{d}\gamma,
\end{aligned}$$

as desired. □

We observe that Sattinger's Theorem (see Sattinger [13, Theorem 5.10]) can now be seen as a particular case of Theorem 2.8, with $W = \mathbf{R}$ under the trivial action of $\Gamma$ in the invariant case and with $(\eta, W) = (\rho, V)$ in the purely equivariant case.

Now, recalling Remark 2.1 and applying the above theorem, we obtain:

**Corollary 2.9.** *Let $\Gamma$ be a compact Lie group. Let $(\rho, V)$ be a finite-dimensional representation of $\Gamma$ with corresponding character $\chi$. Then*

$$\dim \mathcal{Q}^d_V(\Gamma) = \int_\Gamma \sigma(\gamma)\chi_{(d)}(\gamma)\mathrm{d}\gamma$$

*and*

$$\dim \vec{\mathcal{Q}}^d_V(\Gamma) = \int_\Gamma \sigma(\gamma)\chi_{(d)}(\gamma)\chi(\gamma)\mathrm{d}\gamma,$$

*where $\chi_{(d)}$ is the character afforded by the induced action of $\Gamma$ on $S^d V$.*

In order to evaluate these character formulae it is necessary to compute the character $\chi_{(d)}$ of $S^dV$. There is a well-known recursive formula, whose proof can be found for example in Antoneli et al. [1, Section 4]:

$$d\,\chi_{(d)}(\gamma) = \sum_{i=0}^{d-1} \chi(\gamma^{d-i})\chi_{(i)}(\gamma), \tag{2.17}$$

where $\chi_{(0)} = 1$. Also, see Section 6 therein for several examples of calculations using (2.17).

We now recall the Fubini-type theorem that gives the Haar integral over a compact Lie group as an iteration of integrals (see Bröcker and tom Dieck [4, Proposition I 5.16]). Applying the theorem to the character formulae, we get the following useful integral expressions for the dimensions of the spaces of invariants, anti-invariants, equivariants and reversible-equivariants: for an arbitrary (and fixed) $\delta \in \Gamma_-$,

$$\dim \mathcal{P}_V^d(\Gamma) = \frac{1}{2}\left[\int_{\Gamma_+} \chi_{(d)}(\gamma)\mathrm{d}\gamma + \int_{\Gamma_+} \chi_{(d)}(\delta\gamma)\mathrm{d}\gamma\right],$$
$$\dim \mathcal{Q}_V^d(\Gamma) = \frac{1}{2}\left[\int_{\Gamma_+} \chi_{(d)}(\gamma)\mathrm{d}\gamma - \int_{\Gamma_+} \chi_{(d)}(\delta\gamma)\mathrm{d}\gamma\right],$$
$$\dim \vec{\mathcal{P}}_V^d(\Gamma) = \frac{1}{2}\left[\int_{\Gamma_+} \chi_{(d)}(\gamma)\chi_V(\gamma)\mathrm{d}\gamma + \int_{\Gamma_+} \chi_{(d)}(\delta\gamma)\chi_V(\delta\gamma)\mathrm{d}\gamma\right],$$
$$\dim \vec{\mathcal{Q}}_V^d(\Gamma) = \frac{1}{2}\left[\int_{\Gamma_+} \chi_{(d)}(\gamma)\chi_V(\gamma)\mathrm{d}\gamma - \int_{\Gamma_+} \chi_{(d)}(\delta\gamma)\chi_V(\delta\gamma)\mathrm{d}\gamma\right].$$

From these, we obtain

$$\dim \mathcal{P}_V^d(\Gamma) + \dim \mathcal{Q}_V^d(\Gamma) = \int_{\Gamma_+} \chi_{(d)}(\gamma)\mathrm{d}\gamma = \dim \mathcal{P}_V^d(\Gamma_+) \tag{2.18}$$

and

$$\dim \vec{\mathcal{P}}_V^d(\Gamma) + \dim \vec{\mathcal{Q}}_V^d(\Gamma) = \int_{\Gamma_+} \chi_{(d)}(\gamma)\chi_V(\gamma)\mathrm{d}\gamma = \dim \vec{\mathcal{P}}_V^d(\Gamma_+), \tag{2.19}$$

in agreement with Corollaries 2.4 and 2.6.

Next we present two necessary conditions for a representation $(\rho, V)$ of $\Gamma$ to be self-dual.

**Proposition 2.10.** *Let $\Gamma$ be a reversing symmetry group. If $V$ is self-dual, then every reversing symmetry has vanishing trace.*

**Proof.** For $\sigma$ as in (1.3), we have, for every $\gamma \in \Gamma$,

$$\mathrm{tr}(\rho(\gamma)) = \mathrm{tr}(\rho_\sigma(\gamma)) = \mathrm{tr}(\sigma(\gamma)\rho(\gamma)) = \sigma(\gamma)\,\mathrm{tr}(\rho(\gamma)).$$

If $\gamma \in \Gamma_-$, then $\mathrm{tr}(\rho(\gamma)) = -\,\mathrm{tr}(\rho(\gamma))$, that is, $\mathrm{tr}(\rho(\gamma)) = 0$. □

**Corollary 2.11.** *Let $\Gamma$ be a reversing symmetry group. If $V$ is self-dual, then the Hilbert–Poincaré series of $\vec{\mathcal{P}}_V(\Gamma)$ and $\vec{\mathcal{Q}}_V(\Gamma)$ are equal. Moreover, every coefficient of the Hilbert–Poincaré series of $\vec{\mathcal{P}}_V(\Gamma_+)$ is even.*

**Proof.** By Proposition 2.10, we have $\chi_V(\gamma) = 0$ for all $\gamma \in \Gamma_-$. Hence,

$$\dim \vec{\mathcal{P}}_V^d(\Gamma) = \dim \vec{\mathcal{Q}}_V^d(\Gamma),$$

for every $d \in \mathbf{N}$, and so the Hilbert–Poincaré series of $\vec{\mathcal{P}}_V(\Gamma)$ and $\vec{\mathcal{Q}}_V(\Gamma)$ coincide. Moreover, the equality above together with (2.19) implies immediately that $\dim \vec{\mathcal{P}}_V^d(\Gamma_+)$ is even for every $d \in \mathbf{N}$. □

**Remark 2.12.** If $V$ is self-dual, then from Proposition 2.10 and (2.17) it follows that $\chi_{(d)}(\gamma) = 0$ for all $\gamma \in \Gamma_-$ whenever $d$ is odd. Therefore, by a similar argument as applied above to the equivariants and reversible-equivariants, we conclude that

$$\dim \mathcal{P}_V^d(\Gamma) = \dim \mathcal{Q}_V^d(\Gamma), \quad \text{if } d \text{ is odd}.$$

From (2.18), $\dim \mathcal{P}_V^d(\Gamma_+)$ is even whenever $d$ is odd. ◇

We finish this section with some examples.

**Example 2.13** ($\Gamma = \mathbf{S}_2$). Consider the action of the group $\Gamma = \mathbf{S}_2$ on $\mathbf{R}^2$ by permutation of the coordinates. This action is generated by the matrix $\delta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, which we take to be a reversing symmetry. So $\Gamma_+ = \{\mathbf{1}\}$ and $\Gamma_- = \{\delta\}$. Hilbert bases for $\mathcal{P}_{\mathbf{R}^2}(\mathbf{S}_2)$ and $\mathcal{P}_{\mathbf{R}^2}$ are $\{x + y, xy\}$ and $\{x, y\}$, respectively. From (2.18) we have that

$$\dim \mathcal{Q}^1_{\mathbf{R}^2}(\mathbf{S}_2) = \dim \mathcal{Q}^2_{\mathbf{R}^2}(\mathbf{S}_2) = 1, \qquad \dim \mathcal{Q}^3_{\mathbf{R}^2}(\mathbf{S}_2) = \dim \mathcal{Q}^4_{\mathbf{R}^2}(\mathbf{S}_2) = 2.$$

Note that $x^n - y^n$ are anti-invariants for all $n \in \mathbf{N}$. From the identity

$$x^n - y^n = (x + y)(x^{n-1} - y^{n-1}) - xy(x^{n-2} - y^{n-2})$$

and cumbersome calculations it is possible to show that $\mathcal{Q}_{\mathbf{R}^2}(\mathbf{S}_2)$ is generated by $u = x - y$. However, we delay this proof to the next section by a simple and direct use of Theorem 3.1.

By the Molien formula for the equivariants (Gatermann [6, Theorem 12.2]) the Hilbert–Poincaré series of $\vec{\mathcal{P}}_{\mathbf{R}^2}(\mathbf{S}_2)$ is $\Psi^{\Gamma}_{\mathbf{R}^2}(t) = \frac{1}{(1-t)^2}$. Furthermore, it is easy to check that

$$L = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

is the matrix of an $\mathbf{S}_2$-reversible-equivariant linear isomorphism and so this representation of $\mathbf{S}_2$ on $\mathbf{R}^2$ is self-dual. Therefore, by Corollary 2.11, the Hilbert–Poincaré series of $\vec{\mathcal{Q}}_{\mathbf{R}^2}(\mathbf{S}_2)$ is $\tilde{\Psi}^{\Gamma}_{\mathbf{R}^2}(t) = \Psi^{\Gamma}_{\mathbf{R}^2}(t) = \frac{1}{(1-t)^2}$. $\diamond$

**Example 2.14** ($\Gamma = \mathbb{O}$). Consider the action of the octahedral group $\mathbb{O}$ on $\mathbb{R}^3$ generated by

$$\kappa_x = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad S_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad S_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

Here, $\kappa_x$ is the reflection on the plane $x = 0$ and $S_x, S_y$ are the rotations of $\pi/2$ about the axes $x$ and $y$, respectively. Let $\delta = \kappa_x$ act as reversing symmetry and $S_x, S_y$ act as symmetries. Hence $\Gamma_+ = \langle S_x, S_y \rangle \cong \mathbf{S}_4$ and $\Gamma_- = \kappa_x \Gamma_+$. Using the software Singular to compute the generators of $\mathcal{P}_{\mathbf{R}^3}(\mathbf{S}_4)$ we find

$$u_1 = x^2 + y^2 + z^2, \qquad u_2 = x^2y^2 + y^2z^2 + x^2z^2, \qquad u_3 = x^2y^2z^2,$$
$$u_4 = x^3(yz^5 - y^5z) + y^3(x^5z - xz^5) + z^3(xy^5 - x^5y).$$

By Melbourne [11, Lemma A.1], $u_1$, $u_2$ and $u_3$ generate $\mathcal{P}_{\mathbf{R}^3}(\mathbb{O})$. Note that $u_4$ is a homogeneous $\mathbb{O}$-anti-invariant (of degree 9) and, from (2.18), that

$$\dim \mathcal{Q}^i_{\mathbf{R}^3}(\mathbb{O}) = 0, \quad \forall\, i = 1, \ldots, 8 \quad \text{and} \quad \dim \mathcal{Q}^9_{\mathbf{R}^3}(\mathbb{O}) = 1.$$

Hence, it follows that $\mathcal{Q}_{\mathbf{R}^3}(\mathbb{O})$ admits only one generator of degree 9, which can be taken to be $u_4$. Finally, using the software GAP to compute the Hilbert–Poincaré series of $\vec{\mathcal{P}}_{\mathbf{R}^3}(\mathbf{S}_4)$ we obtain

$$\begin{aligned} \tilde{\Psi}^{\mathbf{S}_4}_{\mathbf{R}^3}(t) &= \frac{t - t^2 + t^3}{(1-t)(1-t^2)(1-t^4)} \\ &= t + 2t^3 + t^4 + 4t^5 + 2t^6 + 6t^7 + 4t^8 + \cdots. \end{aligned}$$

Since $\dim \vec{\mathcal{P}}^1_{\mathbf{R}^3}(\mathbf{S}_4) = 1$ is odd, it follows from Corollary 2.11 that such a representation of the octahedral group is non-self-dual. $\diamond$

## 3. The algorithm

In this section we prove the second main result of this paper, namely the algorithmic way to compute anti-invariants and reversible-equivariants. The basic idea is to take advantage of the direct sum decompositions from Corollaries 2.4 and 2.6 to transfer the appropriate basis and generating sets from one ring or module to another. We finish this section by applying these results to some important examples.

### 3.1. Generators of anti-invariants

Based on the direct sum decomposition into two $\mathcal{P}_V(\Gamma)$-modules given by Corollary 2.4, in this subsection we show how to obtain generating sets of $\mathcal{Q}_V(\Gamma)$ and $\mathcal{P}_V(\Gamma_+)$ as modules over $\mathcal{P}_V(\Gamma)$ from a Hilbert basis of $\mathcal{P}_V(\Gamma_+)$.

**Theorem 3.1.** *Let $\Gamma$ be a compact Lie group acting on $V$. Let $\{u_1, \ldots, u_s\}$ be a Hilbert basis of the ring $\mathcal{P}_V(\Gamma_+)$. Set*

$$\tilde{u}_j = S^{\Gamma}_{\Gamma_+}(u_j).$$

*Then $\{\tilde{u}_1, \ldots, \tilde{u}_s\}$ is a generating set of the module $\mathcal{Q}_V(\Gamma)$ over $\mathcal{P}_V(\Gamma)$.*

**Proof.** We need to show that every polynomial function $\tilde{f} \in \mathcal{Q}_V(\Gamma)$ can be written as

$$\tilde{f}(x) = \sum_{j=1}^{s} p_j(x)\tilde{u}_j(x), \quad \forall\, x \in V,$$

where $p_j \in \mathcal{P}_V(\Gamma)$ and $\tilde{u}_j = S^{\Gamma}_{\Gamma_+}(u_j)$. We prove this by induction on the cardinality $s$ of the set $\{u_1, \ldots, u_s\}$.

Let us fix $\delta \in \Gamma_-$ and let $\{u_1, \ldots, u_s\}$ be a Hilbert basis of $\mathcal{P}_V(\Gamma_+)$. From Proposition 2.3, there exists $f \in \mathcal{P}_V(\Gamma_+)$ such that $S^{\Gamma}_{\Gamma_+}(f) = \tilde{f}$. First we write

$$f(x) = \sum_{\alpha_1,\ldots,\alpha_s} a_{\alpha_1\ldots\alpha_s}\, u_1^{\alpha_1}(x) \ldots u_s^{\alpha_s}(x),$$

where $a_{\alpha_1\ldots\alpha_s} \in \mathbf{R}$. Using multi-index notation, we rewrite $f$ as

$$f(x) = \sum_{\alpha} a_\alpha\, u^\alpha(x),$$

with $a_\alpha \in \mathbf{R}$, where $\alpha = (\alpha_1, \ldots, \alpha_s)$, and $u^\alpha = u_1^{\alpha_1} \ldots u_s^{\alpha_s}$. We now compute $\tilde{f} = S^{\Gamma}_{\Gamma_+}(f)$ to get

$$\tilde{f}(x) = \sum_{\alpha} a_\alpha\, (u^\alpha(x) - u^\alpha(\delta x)).$$

- Assume $s = 1$. Then we may write

$$\tilde{f}(x) = \sum_{i} a_i\, (u^i(x) - u^i(\delta x)), \tag{3.1}$$

where $a_i \in \mathbf{R}$ and $i \in \mathbf{N}$. Recall the well-known polynomial identity

$$u^i(x) - u^i(\delta x) = (u(x) - u(\delta x)) \left( \sum_{j=0}^{i-1} u^j(x) u^{(i-1)-j}(\delta x) \right)$$

from which we extract the following polynomial:

$$p_i(x) = \sum_{j=0}^{i-1} u^j(x) u^{(i-1)-j}(\delta x).$$

We now show that $p_i \in \mathcal{P}_V(\Gamma)$. Since $\Gamma_+$ is a normal subgroup of $\Gamma$, it follows that for every $\gamma \in \Gamma_+$ there exists $\tilde{\gamma} \in \Gamma_+$ such that $u(\delta\gamma x) = u(\tilde{\gamma}\delta x) = u(\delta x)$, $\forall\, x \in V$. Thus, for all $\gamma \in \Gamma_+$, we have

$$p_i(\gamma x) = \sum_{j=0}^{i-1} u^j(\gamma x)\, u^{(i-1)-j}(\delta\gamma x) = \sum_{j=0}^{i-1} u^j(x)\, u^{(i-1)-j}(\delta x) = p_i(x).$$

Furthermore, since $\delta^2 \in \Gamma_+$,

$$p_i(\delta x) = \sum_{j=0}^{i-1} u^j(\delta x)\, u^{(i-1)-j}(\delta^2 x) = \sum_{j=0}^{i-1} u^j(\delta x)\, u^{(i-1)-j}(x) = p_i(x).$$

Therefore, (3.1) becomes

$$\tilde{f}(x) = \sum_{i} a_i\, p_i(x)(u(x) - u(\delta x)) = p(x)\tilde{u}(x),$$

where $p = 2\sum_i a_i\, p_i \in \mathcal{P}_V(\Gamma)$.

- Assume that for all sets $\{u_1, \ldots, u_\ell\}$ with $1 \leqslant \ell \leqslant s$ we have

$$\sum_{\alpha} a_\alpha\, (u^\alpha(x) - u^\alpha(\delta x)) = \sum_{j=1}^{\ell} p_j(x)\tilde{u}_j(x), \tag{3.2}$$

where $p_j \in \mathcal{P}_V(\Gamma)$ and $\alpha \in \mathbf{N}^\ell$.

Now consider the set $\{u_1, \ldots, u_\ell\} \cup \{u_{\ell+1}\}$ and let $\tilde{f} \in \mathcal{Q}_V(\Gamma)$. Then we may write

$$\begin{aligned}\tilde{f}(x) &= \sum_{\alpha,\alpha_{\ell+1}} a_{\alpha,\alpha_{\ell+1}}(u^\alpha(x)u_{\ell+1}^{\alpha_{\ell+1}}(x) - u^\alpha(\delta x)u_{\ell+1}^{\alpha_{\ell+1}}(\delta x))\\ &= \sum_{\alpha,\alpha_{\ell+1}} a_{\alpha,\alpha_{\ell+1}}\left[u^\alpha(x)(u_{\ell+1}^{\alpha_{\ell+1}}(x) - u_{\ell+1}^{\alpha_{\ell+1}}(\delta x)) + u_{\ell+1}^{\alpha_{\ell+1}}(\delta x)(u^\alpha(x) - u^\alpha(\delta x))\right],\end{aligned}$$

with $u^\alpha,\ u_{\ell+1}^{\alpha_{\ell+1}} \in \mathcal{P}_V(\Gamma_+)$, $\alpha \in \mathbf{N}^\ell$ and $\alpha_{\ell+1} \in \mathbf{N}$. We use (2.9) to write

$$u^\alpha(x) = v_\alpha(x) + w_\alpha(x) \quad \text{and} \quad u_{\ell+1}^{\alpha_{\ell+1}}(x) = v_{\alpha_{\ell+1}}(x) + w_{\alpha_{\ell+1}}(x),$$

with $v_\alpha,\ v_{\alpha_{\ell+1}} \in \mathcal{P}_V(\Gamma)$ and $w_\alpha,\ w_{\alpha_{\ell+1}} \in \mathcal{Q}_V(\Gamma)$. Then

$$u^\alpha(\delta x) = v_\alpha(x) - w_\alpha(x) \quad \text{and} \quad u_{\ell+1}^{\alpha_{n+1}}(\delta x) = v_{\alpha_{\ell+1}}(x) - w_{\alpha_{\ell+1}}(x).$$

By subtracting, we get

$$u^\alpha(x) - u^\alpha(\delta x) = 2w_\alpha(x) \quad \text{and} \quad u_{\ell+1}^{\alpha_{\ell+1}}(x) - u_{\ell+1}^{\alpha_{\ell+1}}(\delta x) = 2w_{\alpha_{\ell+1}}(x).$$

Therefore,

$$\begin{aligned}\tilde{f}(x) &= \sum_{\alpha,\alpha_{\ell+1}} a_{\alpha,\alpha_{\ell+1}}\left[(v_\alpha(x) + w_\alpha(x))2w_{\alpha_{\ell+1}}(x) + (v_{\alpha_{\ell+1}}(x) - w_{\alpha_{\ell+1}}(x))2w_\alpha(x)\right]\\ &= \sum_{\alpha,\alpha_{\ell+1}} a_{\alpha,\alpha_{\ell+1}}(2v_\alpha(x)w_{\alpha_{\ell+1}}(x) + 2v_{\alpha_{\ell+1}}(x)w_\alpha(x))\\ &= \sum_{\alpha,\alpha_{\ell+1}} a_{\alpha,\alpha_{\ell+1}}\left[v_\alpha(x)(u_{\ell+1}^{\alpha_{\ell+1}}(x) - u_{\ell+1}^{\alpha_{\ell+1}}(\delta x)) + v_{\alpha_{\ell+1}}(x)(u^\alpha(x) - u^\alpha(\delta x))\right]\\ &= \sum_{\alpha,\alpha_{\ell+1}} a_{\alpha,\alpha_{\ell+1}}v_\alpha(x)(u_{\ell+1}^{\alpha_{\ell+1}}(x) - u_{\ell+1}^{\alpha_{\ell+1}}(\delta x)) + \sum_{\alpha_{\ell+1}} v_{\alpha_{\ell+1}}(x)\left(\sum_\alpha a_{\alpha,\alpha_{\ell+1}}(u^\alpha(x) - u^\alpha(\delta x))\right).\end{aligned}$$

By the induction hypothesis (3.2) we can write

$$(u_{\ell+1}^{\alpha_{\ell+1}}(x) - u_{\ell+1}^{\alpha_{\ell+1}}(\delta x)) = p_{\ell+1,\alpha_{\ell+1}}(x)\tilde{u}_{\ell+1}(x)$$

and

$$\sum_\alpha a_{\alpha,\alpha_{\ell+1}}(u^\alpha(x) - u^\alpha(\delta x)) = \sum_{j=1}^{\ell} p_{j,\alpha_{\ell+1}}(x)\tilde{u}_j(x),$$

with $p_{\ell+1,\alpha_{\ell+1}}, p_{j,\alpha_{\ell+1}} \in \mathcal{P}_V(\Gamma)$. Then

$$\begin{aligned}\tilde{f}(x) &= \sum_{\alpha,\alpha_{\ell+1}} a_{\alpha,\alpha_{\ell+1}}v_\alpha(x)(p_{\ell+1,\alpha_{\ell+1}}(x)\tilde{u}_{\ell+1}(x)) + \sum_{\alpha_{\ell+1}} v_{\alpha_{\ell+1}}(x)\left(\sum_{j=1}^{\ell} p_{j,\alpha_{\ell+1}}(x)\tilde{u}_j(x)\right)\\ &= \left(\sum_{\alpha,\alpha_{\ell+1}} a_{\alpha,\alpha_{\ell+1}}v_\alpha(x)p_{\ell+1,\alpha_{\ell+1}}(x)\right)\tilde{u}_{\ell+1}(x) + \sum_{j=1}^{\ell}\left(\sum_{\alpha_{\ell+1}} v_{\alpha_{\ell+1}}(x)p_{j,\alpha_{\ell+1}}(x)\right)\tilde{u}_j(x).\end{aligned}$$

But now we observe that

$$p_{\ell+1} = \sum_{\alpha,\alpha_{\ell+1}} a_{\alpha,\alpha_{\ell+1}}v_\alpha\, p_{\ell+1,\alpha_{\ell+1}} \ \in \mathcal{P}_V(\Gamma)$$

and

$$p_j = \sum_{\alpha_{\ell+1}} v_{\alpha_{\ell+1}}\, p_{j,\alpha_{\ell+1}} \ \in \mathcal{P}_V(\Gamma).$$

Therefore,

$$\tilde{f}(x) = \sum_{j=1}^{\ell+1} p_j(x)\tilde{u}_j(x). \quad \square$$

**Example 3.2** (*Continuation of Examples 2.13 and 2.14*). It is a straightforward consequence of the theorem above that the $\mathbf{S}_2$-anti-invariant $u = x - y$ in Example 2.13 generates $\mathcal{Q}_{\mathbf{R}^2}(\mathbf{S}_2)$ as a $\mathcal{P}_{\mathbf{R}^2}(\mathbf{S}_2)$-module. Also, the $\mathbb{O}$-anti-invariant

$$u_4 = x^3(yz^5 - y^5z) + y^3(x^5z - xz^5) + z^3(xy^5 - x^5y)$$

in Example 2.14 generates $\mathcal{Q}_{\mathbf{R}^3}(\mathbb{O})$ as a $\mathcal{P}_{\mathbf{R}^3}(\mathbb{O})$-module. $\diamond$

Here is the result that gives a generating set of the module $\mathcal{P}_V(\Gamma_+)$ over the ring $\mathcal{P}_V(\Gamma)$:

**Corollary 3.3.** *Let $\Gamma$ be a compact Lie group acting on $V$. Let $\{u_1, \ldots, u_s\}$ be a Hilbert basis of the ring $\mathcal{P}_V(\Gamma_+)$. Set $\tilde{u}_i = S^{\Gamma}_{\Gamma_+}(u_i)$, for $i = 1, \ldots, s$. Then $\{1, \tilde{u}_1, \ldots, \tilde{u}_s\}$ is a generating set of the module $\mathcal{P}_V(\Gamma_+)$ over $\mathcal{P}_V(\Gamma)$.*

### *3.2. Algorithm for computing the reversible-equivariants*

Consider the direct sum decomposition into two $\mathcal{P}_V(\Gamma)$-modules of Corollary 2.6. Based on that, we show in this subsection how to obtain generating sets of $\vec{\mathcal{P}}_V(\Gamma_+)$ and $\vec{\mathcal{Q}}_V(\Gamma)$ as modules over $\mathcal{P}_V(\Gamma)$ from a Hilbert basis of $\mathcal{P}_V(\Gamma_+)$ together with a generating set of $\vec{\mathcal{P}}_V(\Gamma_+)$ as a module over $\mathcal{P}_V(\Gamma_+)$. In particular, we achieve our ultimate goal, which is to show how the construction of a generating set for the module $\vec{\mathcal{Q}}_V(\Gamma)$ of reversible-equivariant polynomial mappings over $\mathcal{P}_V(\Gamma)$ can be reduced to a problem in standard invariant theory, whose solution is well known in several important cases.

**Lemma 3.4.** *Let $\Gamma$ be a compact Lie group acting on $V$. Let $\{u_1, \ldots, u_s\}$ be a Hilbert basis of $\mathcal{P}_V(\Gamma_+)$. Let $\{\tilde{u}_0 \equiv 1, \tilde{u}_1, \ldots, \tilde{u}_s\}$ be the generating set of the module $\mathcal{P}_V(\Gamma_+)$ over the ring $\mathcal{P}_V(\Gamma)$ obtained from $\{u_1, \ldots, u_s\}$ as in Corollary 3.3 and $\{H_0, \ldots, H_r\}$ a generating set of the module $\vec{\mathcal{P}}_V(\Gamma_+)$ over the ring $\mathcal{P}_V(\Gamma_+)$. Then*

$$\{H_{ij} = \tilde{u}_i H_j : i = 0, \ldots, s;\ j = 0, \ldots, r\}$$

*is a generating set of the module $\vec{\mathcal{P}}_V(\Gamma_+)$ over the ring $\mathcal{P}_V(\Gamma)$.*

**Proof.** Let $G \in \vec{\mathcal{P}}_V(\Gamma_+)$. Then

$$G = \sum_{j=0}^{r} p_j H_j, \tag{3.3}$$

with $p_j \in \mathcal{P}_V(\Gamma_+)$. Since $\{\tilde{u}_0, \ldots, \tilde{u}_s\}$ is a generating set of the module $\mathcal{P}_V(\Gamma_+)$ over the ring $\mathcal{P}_V(\Gamma)$, it follows that

$$p_j = \sum_{i=0}^{s} p_{ij} \tilde{u}_i, \tag{3.4}$$

with $p_{ij} \in \mathcal{P}_V(\Gamma)$. From (3.3) and (3.4) we get

$$G = \sum_{j=0}^{r} \left( \sum_{i=0}^{s} p_{ij} \tilde{u}_i \right) H_j = \sum_{i,j=0}^{s,r} p_{ij} \left(\tilde{u}_i H_j\right),$$

as desired. $\square$

**Lemma 3.5.** *Let $\Gamma$ be a compact Lie group acting on $V$. Let $\{H_{00}, \ldots, H_{sr}\}$ be a generating set of the module $\vec{\mathcal{P}}_V(\Gamma_+)$ over the ring $\mathcal{P}_V(\Gamma)$ given as in Lemma 3.4. Then*

$$\{\tilde{H}_{ij} = \vec{S}^{\Gamma}_{\Gamma_+}(H_{ij}) : i = 0, \ldots, s;\ j = 0, \ldots, r\}$$

*is a generating set of the module $\vec{\mathcal{Q}}_V(\Gamma)$ over the ring $\mathcal{P}_V(\Gamma)$.*

**Proof.** Let $\{H_{00}, \ldots, H_{sr}\}$ be a generating set of the $\mathcal{P}_V(\Gamma)$-module $\vec{\mathcal{P}}_V(\Gamma_+)$ and let $\tilde{G} \in \vec{\mathcal{Q}}_V(\Gamma)$. From Proposition 2.5, there exists $G \in \vec{\mathcal{P}}_V(\Gamma_+)$ such that $\tilde{G} = \vec{S}^{\Gamma}_{\Gamma_+}(G)$. Now write

$$G = \sum_{i,j=0}^{s,r} p_{ij} H_{ij}$$

with $p_{ij} \in \mathcal{P}_V(\Gamma)$. Applying $\vec{S}^{\Gamma}_{\Gamma_+}$ on both sides, we get

$$\tilde{G} = \sum_{i,j=0}^{s,r} p_{ij} \vec{S}^{\Gamma}_{\Gamma_+}(H_{ij}) = \sum_{i,j=0}^{s,r} p_{ij} \tilde{H}_{ij},$$

as desired. $\square$

It is now immediate from the two lemmas above the following result:

**Theorem 3.6.** *Let $\Gamma$ be a compact Lie group acting on $V$. Let $\{u_1, \ldots, u_s\}$ be a Hilbert basis of $\mathcal{P}_V(\Gamma_+)$ and $\{H_0, \ldots, H_r\}$ be a generating set of the module $\vec{\mathcal{P}}_V(\Gamma_+)$ over the ring $\mathcal{P}_V(\Gamma_+)$. Let $\{\tilde{u}_0 \equiv 1, \tilde{u}_1, \ldots, \tilde{u}_s\}$ be the generating set of the module $\mathcal{P}_V(\Gamma_+)$ over the ring $\mathcal{P}_V(\Gamma)$ obtained from $\{u_1, \ldots, u_s\}$ as in Corollary 3.3. Then*

$$\{\tilde{H}_{ij} = \vec{S}^{\Gamma}_{\Gamma_+}(\tilde{u}_i H_j) : i = 0, \ldots, s\,;\ j = 0, \ldots, r\}$$

*is a generating set of the module $\vec{\mathcal{Q}}_V(\Gamma)$ over $\mathcal{P}_V(\Gamma)$.*

Here is the procedure to find a generating set of the module $\vec{\mathcal{Q}}_V(\Gamma)$ over the ring $\mathcal{P}_V(\Gamma)$:

**Algorithm 3.7** (*Generating Set of Reversible-Equivariants*).

INPUT:
- · compact Lie group $\Gamma \subset \mathbf{O}(n)$
- · normal subgroup $\Gamma_+ \subset \Gamma$ of index 2
- · $\delta \in \Gamma \setminus \Gamma_+$
- · Hilbert basis $\{u_1, \ldots, u_s\}$ of $\mathcal{P}_V(\Gamma_+)$
- · generating set $\{H_0, \ldots, H_r\}$ of $\vec{\mathcal{P}}_V(\Gamma_+)$ over $\mathcal{P}_V(\Gamma_+)$

OUTPUT: generating set $\{\tilde{K}_1, \ldots, \tilde{K}_\ell\}$ of the module of reversible-equivariants $\vec{\mathcal{Q}}_V(\Gamma)$ over the ring $\mathcal{P}_V(\Gamma)$

PROCEDURE:
```
k := 1
for i from 1 to s do
    ũ_i(x) := 1/2 (u_i(x) − u_i(δx))
    for j from 0 to r do
        H_0j(x) := H_j(x)
        H_ij(x) := ũ_i(x) H_j(x)
        H̃_ij(x) := 1/2 (H_ij(x) − δ^{-1} H_ij(δx))
        if H̃_ij ≠ 0 then
            K̃_k := H̃_ij
            k := k + 1
        end
    end
end
l := k − 1
return {K̃_1, ..., K̃_ℓ}
```

The number $\ell \leq rs$ that appears in the algorithm output is the resulting number of non-zero generating polynomials for $\vec{\mathcal{Q}}_V(\Gamma)$.

### 3.3. Examples

We now illustrate the method with some examples. We reproduce all the steps of Algorithm 3.7 for the first example. We omit the steps for the others, presenting the output (generators of the reversible-equivariants) and also the anti-invariants computed by the algorithm. It is worthwhile to point out here that, despite our method being general, it is necessary only when the representation at hand is non-self-dual. Otherwise, when the representation is self-dual, there is a much more efficient procedure which takes advantage of the existence of a reversible-equivariant linear isomorphism $L : V \to V$ to obtain the generators of $\vec{\mathcal{Q}}_V(\Gamma)$ from those of $\vec{\mathcal{P}}_V(\Gamma)$ via the "pull-back" of $L$. See Baptistelli and Manoel [3] for details.

**Example 3.8** ($\Gamma = \mathbf{D}_n$). Consider the action of the dihedral group $\Gamma = \mathbf{D}_n$ on $\mathbf{C}$ generated by the complex conjugation $\kappa$ and the rotation $R_{2\pi/n}$:

$$\kappa z = \bar{z} \quad \text{and} \quad R_{\frac{2\pi}{n}} z = \mathrm{e}^{\mathrm{i}\frac{2\pi}{n}} z.$$

Let us take $\delta = R_{2\pi/n} \in \Gamma_-$ and $\kappa \in \Gamma_+$. We observe that this is only possible for $n$ even (see [2, p. 249]). Then

$$\Gamma_+ = \langle \kappa, R^2_{\frac{2\pi}{n}} \rangle = \mathbf{D}_{\frac{n}{2}}.$$

It is well known (Golubitsky et al. [8, Section XII 4]) that

$$u_1(z) = z\bar{z} \quad \text{and} \quad u_2(z) = z^{\frac{n}{2}} + \bar{z}^{\frac{n}{2}}$$

constitute a Hilbert basis for $\mathcal{P}_{\mathbf{C}}(\mathbf{D}_{n/2})$. Also, from Golubitsky et al. [8, Section XII 5],

$$H_0(z) = z \quad \text{and} \quad H_1(z) = \bar{z}^{\frac{n}{2}-1}.$$

are generators of $\vec{\mathcal{P}}(\mathbf{D}_{n/2})$. The algorithm then computes

$$\begin{aligned}
\tilde{u}_1(z) &= \frac{1}{2}(u_1(z) - u_1(\mathrm{e}^{\mathrm{i}\frac{2\pi}{n}} z)) = 0\\
\tilde{u}_2(z) &= \frac{1}{2}(u_2(z) - u_2(\mathrm{e}^{\mathrm{i}\frac{2\pi}{n}} z)) = z^{\frac{n}{2}} + \bar{z}^{\frac{n}{2}},\\
H_{00}(z) &= H_0(z) = z,\\
H_{01}(z) &= H_1(z) = \bar{z}^{\frac{n}{2}-1},\\
H_{10}(z) &= \tilde{u}_1(z)H_0(z) = 0,\\
H_{11}(z) &= \tilde{u}_1(z)H_1(z) = 0,\\
H_{20}(z) &= \tilde{u}_2(z)H_0(z) = z^{\frac{n}{2}+1} + (z\bar{z})\bar{z}^{\frac{n}{2}-1},\\
H_{21}(z) &= \tilde{u}_2(z)H_1(z) = (z\bar{z})^{\frac{n}{2}-1}z + \bar{z}^{n-1}
\end{aligned}$$

and

$$\begin{aligned}
\tilde{H}_{00} &= \tilde{H}_{21} \equiv 0,\\
\tilde{H}_{01}(z) &= \bar{z}^{\frac{n}{2}-1},\\
\tilde{H}_{20}(z) &= z^{\frac{n}{2}+1} + (z\bar{z})\bar{z}^{\frac{n}{2}-1}.
\end{aligned}$$

The output is

$$\tilde{K}_1 = \bar{z}^{\frac{n}{2}-1}, \qquad \tilde{K}_2 = (z\bar{z})^{\frac{n}{2}-1}z + \bar{z}^{n-1}.$$

By noting that $z\bar{z}$ is a $\mathbf{D}_n$-invariant, it is obvious that $\{\bar{z}^{n/2-1}, z^{n/2+1}\}$ is also a generating set of the module $\vec{\mathcal{Q}}_{\mathbf{C}}(\mathbf{D}_n)$ over the ring $\mathcal{P}_{\mathbf{C}}(\mathbf{D}_n)$. We observe that Baptistelli and Manoel [2, p. 249] use a different approach to deal with this example, obtaining this set of generators by direct calculations. $\diamond$

**Example 3.9** ($\Gamma = \mathbf{Z}_2$). Consider the action of the group $\Gamma = \mathbf{Z}_2$ on $\mathbf{R}^2$ generated by the reflection $\delta = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, which we take to be a reversing symmetry. So $\Gamma_+ = \{\mathbf{1}\}$ and $\Gamma_- = \{\delta\}$. It is clear that $u_1(x, y) = x$ and $u_2(x, y) = y$ constitute a Hilbert basis of $\mathcal{P}_{\mathbf{R}^2}$. Also, $H_0(x, y) = (1, 0)$ and $H_1(x, y) = (0, 1)$ constitute a generating set of the module $\vec{\mathcal{P}}_{\mathbf{R}^2}$ over the ring $\mathcal{P}_{\mathbf{R}^2}$. Algorithm 3.7 produces

$$\tilde{u}_1(x, y) = x, \qquad \tilde{u}_2(x, y) = y$$

and

$$\tilde{K}_1(x, y) = (1, 0), \qquad \tilde{K}_2(x, y) = (0, 1). \quad \diamond$$

**Example 3.10** ($\Gamma = \mathbf{Z}_2 \oplus \mathbf{Z}_2$). Consider the action of the group $\Gamma = \mathbf{Z}_2 \oplus \mathbf{Z}_2$ on $\mathbf{R}^2$ generated by the reflections

$$\kappa_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \kappa_2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Let us take $\delta = \kappa_1 \in \Gamma_-$ and $\kappa_2 \in \Gamma_+$. So $\Gamma_+ = \mathbf{Z}_2(\kappa_2) = \{\mathbf{1}, \kappa_2\}$ and $\Gamma_- = \{\kappa_1, -\mathbf{1}\}$. It is well known that $u_1(x, y) = x^2$ and $u_2(x, y) = y$ constitute a Hilbert basis of $\mathcal{P}_{\mathbf{R}^2}(\mathbf{Z}_2(\kappa_2))$. Also, $H_0(x, y) = (x, 0)$ and $H_1(x, y) = (0, 1)$ constitute a generating set of $\vec{\mathcal{P}}_{\mathbf{R}^2}(\mathbf{Z}_2(\kappa_2))$. By Algorithm 3.7,

$$\tilde{u}_1(x, y) = 0, \qquad \tilde{u}_2(x, y) = y,$$

and

$$\tilde{K}_1(x, y) = (0, 1), \qquad \tilde{K}_2(x, y) = (xy, 0).$$

Observe that if we choose $\Gamma_+ = \mathbf{Z}_2(\kappa_1) = \{\mathbf{1}, \kappa_1\}$ and $\Gamma_- = \{\kappa_2, -\mathbf{1}\}$, then $\mathcal{Q}_{\mathbf{R}^2}(\mathbf{Z}_2 \oplus \mathbf{Z}_2)$ is generated by $\{x\}$ and $\vec{\mathcal{Q}}_{\mathbf{R}^2}(\mathbf{Z}_2 \oplus \mathbf{Z}_2)$ is generated by $\{(1, 0), (0, xy)\}$ over $\mathcal{P}_{\mathbf{R}^2}(\mathbf{Z}_2 \oplus \mathbf{Z}_2)$. $\diamond$

**Example 3.11** ($\Gamma = \mathbf{O}(2)$). Consider the orthogonal group $\mathbf{O}(2)$ acting on $\mathbf{C} \times \mathbf{R}$, where the rotations $\theta \in \mathbf{SO}(2)$ and the flip $\kappa$ act as

$$\theta(z, x) = (\mathrm{e}^{\mathrm{i}\theta}z, x) \quad \text{and} \quad \kappa(z, x) = (\bar{z}, -x).$$

Consider $\Gamma_+ = \mathbf{SO}(2)$ and $\delta = \kappa \in \Gamma_-$. We have that $u_1(z,x) = z\bar{z}$ and $u_2(z,x) = x$ form a Hilbert basis of $\mathcal{P}_{\mathbf{C}\times\mathbf{R}}(\mathbf{SO}(2))$. Also $H_0(z,x) = (\mathrm{i}z, 0)$, $H_1(z,x) = (z, 0)$ and $H_3(z,x) = (0, 1)$ constitute a generating set of $\vec{\mathcal{P}}_{\mathbf{C}\times\mathbf{R}}(\mathbf{SO}(2))$. By Algorithm 3.7,

$$\tilde{u}_1(z,x) = 0, \qquad \tilde{u}_2(z,x) = x$$

and

$$\tilde{K}_1(z,x) = (\mathrm{i}z, 0), \qquad \tilde{K}_2(z,x) = (xz, 0), \qquad \tilde{K}_3(z,x) = (0, 1). \quad \diamond$$

Our last example deals with the group given by the semi-direct product $\mathbf{D}_6 \dotplus \mathbf{T}^2$ summing direct with $\mathbf{Z}_2$. This is the full group of symmetries of the PDE that models the Rayleigh–Bénard convection. The steady-state bifurcation problem of this fluid motion has been treated in Golubitsky et al. [8, Case Study 4]. In this example, the dihedral group $\mathbf{D}_6$ is realized as the group generated by a three-element permutation and by a flip.

**Example 3.12** ($\Gamma = (\mathbf{D}_6 \dotplus \mathbf{T}^2) \oplus \mathbf{Z}_2$). Consider the group $\Gamma = (\mathbf{D}_6 \dotplus \mathbf{T}^2) \oplus \mathbf{Z}_2$ acting on $\mathbf{C}^3$ as follows: for $(z_1, z_2, z_3) \in \mathbf{C}^3$,

(a) the even permutations in $\mathbf{D}_6$ act by permuting the coordinates of $(z_1, z_2, z_3)$,
(b) the flip permutation in $\mathbf{D}_6$ acts as $(z_1, z_2, z_3) \mapsto (\bar{z}_1, \bar{z}_2, \bar{z}_3)$,
(c) $\theta = (\theta_1, \theta_2)$ in the torus $\mathbf{T}^2$ acts as

$$\theta \cdot (z_1, z_2, z_3) = (\mathrm{e}^{\mathrm{i}\theta_1 p} z_1, \mathrm{e}^{\mathrm{i}\theta_2 p} z_2, \mathrm{e}^{-\mathrm{i}(\theta_1+\theta_2)p} z_3),$$

(d) the reflection $\kappa \in \mathbf{Z}_2$ acts as minus the identity:

$$\kappa(z_1, z_2, z_3) = (-z_1, -z_2, -z_3).$$

In this example we consider $\delta = \kappa \in \Gamma_-$ and so $\Gamma_+ = \mathbf{D}_6 \dotplus \mathbf{T}^2$.

Let $v_j = z_j \bar{z}_j$ ($j = 1, 2, 3$) and consider the elementary symmetric polynomials in $v_j$:

$$u_1 = v_1 + v_2 + v_3, \qquad u_2 = v_1 v_2 + v_1 v_3 + v_2 v_3, \qquad u_3 = v_1 v_2 v_3.$$

Also, let $u_4 = z_1 z_2 z_3 + \bar{z}_1 \bar{z}_2 \bar{z}_3$. It is well known that $\{u_1, \ldots, u_4\}$ is a Hilbert basis for $\mathcal{P}_{\mathbf{C}^3}(\mathbf{D}_6 \dotplus \mathbf{T}^2)$ (see Golubitsky et al. [8, Theorem 3.1(a), p. 156]). Also, from Golubitsky et al. [8, Theorem 3.1(b), p. 156], we have that a generating set for $\vec{\mathcal{P}}_{\mathbf{C}^3}(\mathbf{D}_6 \dotplus \mathbf{T}^2)$ is given by

$$\begin{aligned}
H_0(z_1,z_2,z_3) &= (z_1, z_2, z_3),\\
H_1(z_1,z_2,z_3) &= (u_1 z_1, u_2 z_2, u_3 z_3),\\
H_2(z_1,z_2,z_3) &= (u_1^2 z_1, u_2^2 z_2, u_3^2 z_3),\\
H_3(z_1,z_2,z_3) &= (\bar{z}_2\bar{z}_3, \bar{z}_1\bar{z}_3, \bar{z}_1\bar{z}_2),\\
H_4(z_1,z_2,z_3) &= (u_1\bar{z}_2\bar{z}_3, u_2\bar{z}_1\bar{z}_3, u_3\bar{z}_1\bar{z}_2),\\
H_5(z_1,z_2,z_3) &= (u_1^2\bar{z}_2\bar{z}_3, u_2^2\bar{z}_1\bar{z}_3, u_3^2\bar{z}_1\bar{z}_2).
\end{aligned}$$

Algorithm 3.7 gives

$$\tilde{u}_j(z_1,z_2,z_3) = 0 \quad (j = 1, 2, 3), \qquad \tilde{u}_4(z_1,z_2,z_3) = z_1 z_2 z_3 + \bar{z}_1\bar{z}_2\bar{z}_3$$

and

$$\begin{aligned}
\tilde{K}_1(z_1,z_2,z_3) &= (\bar{z}_2\bar{z}_3, \bar{z}_1\bar{z}_3, \bar{z}_1\bar{z}_2),\\
\tilde{K}_2(z_1,z_2,z_3) &= (u_1\bar{z}_2\bar{z}_3, u_2\bar{z}_1\bar{z}_3, u_3\bar{z}_1\bar{z}_2),\\
\tilde{K}_3(z_1,z_2,z_3) &= (u_1^2\bar{z}_2\bar{z}_3, u_2^2\bar{z}_1\bar{z}_3, u_3^2\bar{z}_1\bar{z}_2),\\
\tilde{K}_4(z_1,z_2,z_3) &= (u_4 z_1, u_4 z_2, u_4 z_3),\\
\tilde{K}_5(z_1,z_2,z_3) &= (u_4 u_1 z_1, u_4 u_2 z_2, u_4 u_3 z_3),\\
\tilde{K}_6(z_1,z_2,z_3) &= (u_4 u_1^2 z_1, u_4 u_2^2 z_2, u_4 u_3^2 z_3).
\end{aligned}$$

We observe that the generators of the $((\mathbf{D}_6 \dotplus \mathbf{T}^2) \oplus \mathbf{Z}_2)$-reversible-equivariants turned out to be $u_4 H_0, u_4 H_1, u_4 H_2, H_3, H_4, H_5$, while the $((\mathbf{D}_6 \dotplus \mathbf{T}^2) \oplus \mathbf{Z}_2)$-equivariants are $H_0, H_1, H_2, u_4 H_3, u_4 H_4, u_4 H_5$ (as given in Golubitsky et al. [8, Corollary 3.2, p. 157]). $\diamond$

## Acknowledgments

F. Antoneli was supported by FCT grant SFRH/BPD/34534/2006 and by FAPESP grant 07/03519-6. P.H. Baptistelli was supported by a CNPq grant. P.H. Baptistelli and M. Manoel thank the University of Porto for its hospitality, where part of this work was carried out.

CMUP is supported by FCT through the programmes POCTI and POSI, with Portuguese and European Community structural funds.

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